On irreducibility and disjointness of Koopman and quasi-regular representations of weakly branch groups.

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1 Introduction.

The main resources of examples of unitary representations of a countable group are Koopman, quasi-regular and groupoid representations. In the present paper we will study the Koopman and quasi-regular representations corresponding to actions of weakly branch groups on rooted trees. Their relation to the groupoid representation is studied in the second paper of the authors [12] on the subject of spectral properties.

Branch groups were introduced by the second author in [17] and play important role in many investigations in group theory and around (see [5]). They posses interesting and often unusual properties. Branch just infinite groups constitute one of three classes on which the class of just infinite groups (i.e. infinite groups whose proper quotients are finite) naturally splits. The class of branch groups contains groups of intermediate growth, amenable but not elementary amenable groups, groups with finite commutator width etc.. Weakly branch groups is a natural generalization of the class of branch

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groups and keep many nice properties of branch groups (for instance absence of nontrivial laws, see [1]). Weakly branch groups also play important role in studies in holomorphic dynamics (see [28]) and in the theory of fractals (see [20]).

Throughout this paper we will assume that G is a countable group. To every subgroup H of G one can associate a quasi-regular representation $\rho_{G/H}$ acting in $l^2(G/H)$. In particular, given an action of G on a set X for every $x \in X$ one can consider the corresponding quasi-regular representation $\rho_x = \rho_{G/\mathrm{St}_G(x)}$, where $\mathrm{St}_G(x)$ is the stabilizer of x in G. Spectral proberties of quasi-regular representations play important roles in the Random Walks on groups and Schreier graphs (see e.g. [23], [22] and [18]). In the case when $H = \{e\}$ (i.e. is the trivial subgroup) the quasi-regular representation coincides with the regular representation ρ_G which is of a special importance. Quasi-regular representations naturally give rise to Hecke algebras and their representations (see e.g. [2] and [29]).

A group acting on a rooted tree T is called weakly branch if it acts transitively on each level of the tree and for every vertex v of T it has a non-trivial element g supported on the subtree T_v emerging from v (see e.g. [5] and [18]). Using Mackey criterion of irreducibility of quasi-regular representations Bartholdi and the second author in [4] showed that quasi-regular representations corresponding to the action of a weakly branch group on the boundary of a rooted tree are irreducible. One of the results of the present paper is the following:

Theorem 1. *Let G be a countable weakly branch group acting on a spherically homogeneous rooted tree T. Then for any $x, y \in \partial T$, where ∂T is the boundary of the tree, such that x and y are from disjoint orbits the quasi-regular representations ρ_x and ρ_y are not unitary equivalent.*

In particular, we obtain a continuum of pairwise disjoint (we will use the word "disjoint" as a synonym to "not unitary equivalent") irreducible representations of a weakly branch group G. Using Theorem 1 we show the following:

Theorem 2. *Let G be a countable weakly branch group acting on a spherically homogeneous rooted tree T. Then the centralizer of G in the group $\mathrm{Bij}(\partial T)$ of all bijections of the boundary of T onto itself is trivial.*

This gives triviality of centralizers of G in such groups as $\mathrm{Homeo}(\partial T)$, $\mathrm{Aut}(\partial T)$ etc. (see Corollary 2). As the result, we obtain a new example

of a dynamical system for which the generalized Ismagilov's conjecture fails (see [24]).

Another important family of representations gives the Koopman representation κ associated to a dynamical system (G, X, μ), where (X, μ) is a measure space on which G acts by measure class preserving transformations. Such representations are rarely irreducible. In the case when μ is a G-invariant probability measure the subspace of constant functions in $L^2(X, \mu)$ is $\kappa(G)$-invariant. And moreover, the restriction κ_0 of κ on the subspace $L^2_0(X, \mu) \subset L^2(X, \mu)$ of functions with zero integral (orthogonal complement to constant functions) is also usually reducible. A few known exceptions are listed in [16]. Recently, attention to irreducibility problem of representation κ_0 was raised by Vershik in [32]. One of the results of the present paper is constructing new examples of irreducible Koopman representations corresponding to actions with quasi-invariant measures.

For a d-regular rooted tree T its boundary ∂T can be identified with a space of sequences $\{x_j\}_{j\in\mathbb{N}}$ where $x_j \in \{1, \dots, d\}$. Let

$$\mathcal{P} = \{p = (p_1, p_2, \dots, p_d) : p_i > 0 \ \text{ for } \ i = 1, 2, \dots, d \ \text{ and } \ \sum_{i=1}^{d} p_i = 1\} \quad (1)$$

be the set of all probability distributions on the alphabet $\{1, 2, \dots, d\}$ assigning positive probability to every letter and

$$\mathcal{P}^* = \{p \in \mathcal{P} : p_i \neq p_j \ \text{ for all } \ 1 \leqslant i < j \leqslant d\}. \quad (2)$$

For $p \in \mathcal{P}$ denote by $\mu_p = \prod_{\mathbb{N}} p$ the corresponding Bernoulli measure on ∂T. Our main result is:

Theorem 3. *Let G be a subexponentially bounded countable weakly branch group acting on a regular rooted tree and $p \in \mathcal{P}^*$. Then the following holds:*

1) *the Koopman representation κ_p associated to the action of G on $(\partial T, \mu_p)$ is irreducible;*

2) *this representation is not unitary equivalent to any of the quasi-regular representations $\rho_x, x \in \partial T$;*

3) *Koopman representations associated to different $p \in \mathcal{P}^*$ are pairwise disjoint.*

Here subexponentially bounded group means a group consisting of subexponentially bounded automorphisms of T. The precise definition will be given in Subsection 2.1. Notice that Theorem 3 gives an additional continuum of pairwise disjoint irreducible representations of a weakly branch group. These representations are faithful ($\kappa_p(g) \neq \mathrm{Id}$ if g is not the group unit). Observe also that for the uniform distribution $u = (\frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d}) \in \mathcal{P}$ the corresponding Bernoulli measure μ_u is G-invariant and the Koopman representation κ_u is a direct sum of countably many finite-dimensional representations (see [4]). In fact, the authors see a possibility of generalizing Theorem 3 to the case of distributions $p \in \mathcal{P} \setminus \{u\}$ (i.e. such that $p_i \neq p_j$ for some $1 \leqslant i < j \leqslant d$). However, since this leads to further complications of already difficult proof in this paper we focus on distributions from $\mathcal{P}^*$.

2 Preliminaries.

In this section we give necessary preliminaries on groups acting on rooted trees, representation theory and related topics.

2.1 Weakly branch groups.

Here we give a brief introduction to actions of groups on boundaries of rooted trees. We refer the reader to [18] for detailed definitions and properties of these actions.

A rooted tree is a tree T, with vertex set divided into levels V_n, $n \in \mathbb{Z}_+$, such that V_0 consists of one vertex v_0 (called the root of T), the edges are only between consecutive levels, and each vertex from V_n, $n \geqslant 1$ (we consider infinite trees), is connected by an edge to exactly one vertex from V_{n-1} (and several vertices from V_{n+1}). A rooted tree is called spherically homogeneous if each vertex from V_n connected to the same number d_n of vertices from V_{n+1}. T is called $d-$regular, if $d_n = d$ is the same for all levels.

An automorphism of a rooted tree T is any automorphism of the graph T preserving the root. The boundary ∂T of T is the set of all infinite paths starting at v_0 and passing through each level exactly one time. For a vertex v of T denote by $\partial T_v \subset \partial T$ the set of paths passing through v. Supply ∂T by the topology generated by the sets ∂T_v. Automorphisms of T act naturally on ∂T by homeomorphisms. If T is spherically homogeneous then ∂T admits a unique $\mathrm{Aut}(T)$-invariant measure μ. This measure is uniform in the sense

that

$$\mu(\partial T_v) = \frac{1}{d_0 d_1 \dots d_{n-1}} \quad \text{for any} \;\; n \;\; \text{and any} \;\; v \in V_n.$$

Grigorchuk, Nekrashevich and Suschanski showed that this measure is ergodic if and only if the action of G is transitive on each level V_n of T (level transitive). Moreover, in this case it is uniquely ergodic. We refer the reader to [21], Proposition 6.5 for details.

Definition 1. Let T be a spherically homogeneous tree and $G < \mathrm{Aut}(T)$. Rigid stabilizer of a vertex v is the subgroup $\mathrm{rist}_v(G) = \{g \in G : \mathrm{supp}(g) \subset T_v\}$. Rigid stabilizer of level n is

$$\mathrm{rist}_n(G) = \prod_{v \in V_n} \mathrm{rist}_v(G).$$

G is called *branch* if it is transitive on each level and $\mathrm{rist}_n(G)$ is a subgroup of finite index in G for all n. G is called *weakly branch* if it is transitive on each level V_n of T and $\mathrm{rist}_v(G)$ is nontrivial for each v.

For each level V_n of a binary rooted tree an automorphism g of T can be presented in the form

$$g = \sigma \cdot (g_1, \dots, g_{d^n}),$$

where $\sigma \in \mathrm{Sym}(V_n)$ is a permutation of the vertices from V_n and g_i are the restrictions of g on the subtrees emerging from the vertices of V_n.

Definition 2. An element $g \in \mathrm{Aut}(T)$ is polynomially bounded (in the sense of Sidki [30]) if the number $k_n(g)$ of restrictions g_i to the vertices of level n not equal to identity automorphism is bounded by a polynomial of n. We will call g subexponentially bounded if for every $0 < \gamma < 1$ one has

$$\lim_{n \to \infty} k_n(g) \gamma^n = 0.$$

A group $G < \mathrm{Aut}(T)$ is polynomially (subexponentially) bounded if each $g \in G$ is polynomially (subexponentially) bounded.

R. Kravchenko showed (see [25]) that for any polynomially bounded automorphism g defined by a finite automaton and any $p \in \mathcal{P}$ (see (1)) the measure μ_p is quasi-invariant with respect to the action of g. We will show in Section 4 that in fact the condition that g is defined by a finite automaton is redundant and polynomial boundedness can be weakened to subexponential.

2.2 Quasi-regular representations.

Given a countable group acting on a set X and a point $x \in X$ one can define the quasi-regular representation ρ_x in $l^2(Gx)$, where Gx is the orbit of x, by:

$$(\rho_x(g)f)(y) = f(g^{-1}y).$$

Notice that the isomorphism class of ρ_x depends only on the stabilizer $\mathrm{St}_G(x)$ of x.

Recall that two subgroups H_1, H_2 of a group G are called *commensurable* if $H_1 \cap H_2$ is of finite index in both H_1 and H_2. The groups H_1 and H_2 are called *quasi-conjugate* in G if gH_1g^{-1} is commensurable to H_2 for some $g \in G$. By definition, *commensurator* of $H < G$ is the subgroup of G defined by

$$\mathrm{comm}_G(H) = \{g \in G : H \cap gHg^{-1} \text{ has finite index in } H \text{ and } gHg^{-1}\}.$$

Mackey proved the following (see [27], Corollary 7):

Theorem 4. 1) *Let H be a subgroup of an infinite discrete countable group G. Then the quasi-regular representation $\rho_{G/H}$ is irreducible if and only if* $\mathrm{comm}_G(H) = H$.
2) *Let H_1, H_2 be two subgroups of G such that* $\mathrm{comm}_G(H_i) = H_i$, $i = 1, 2$. *Then ρ_{G/H_1} and ρ_{G/H_2} are unitary equivalent if and only if H_1 and H_2 are quasi-conjugate.*

In [4] the authors showed the following:

Proposition 1. *Let G be a weakly branch group, T be the corresponding spherically homogeneous rooted tree, $x \in \partial T$ and $\mathrm{St}_G(x)$ be its stabilizer. Then* $\mathrm{comm}_G(\mathrm{St}_G(x)) = \mathrm{St}_G(x)$.

Theorem 4 together with Proposition 1 immediately imply:

Corollary 1. *For a weakly branch group G and any $x \in \partial T$ the quasi-regular representation ρ_x is irreducible.*

2.3 Koopman representation.

The most natural representations that one can associate to a measure-preserving action of a group G on a measure space (X, μ), where μ is a

probabilty measure, is the Koopman representation κ of G in $L^2(X,\mu)$ acting by:

$$(\kappa(g)f)(x) = f(g^{-1}x).$$

This representation is important due to the fact that the spectral properties of κ reflect the dynamical properties of the action such as ergodicity and weak-mixing.

It is known that for an ergodic action operators $\kappa(g)$ together with operators of multiplication by functions from $L^\infty(X,\mu)$ generate in the weak operator topology the algebra of all bounded operators on $L^2(X,\mu)$. The representation κ has invariant subspace of constant functions on X. However, it is natural to ask whether κ is irreducible in the orthogonal complement of the constant functions in $L^2(X,\mu)$ (κ is called *almost irreducible* in this case). This question is discussed in E. Glasner's book [16] and is again raised in Vershik's paper [32] (Problem 4). The cases when the answer is positive are quite rare. Among few examples is the example of arbitrary dense subgroup of the group $\mathrm{Aut}([0,1],\mu)$ of all measure preserving automorphisms of the unit segment with Lebesgue measure μ, supplied by the weak topology (see [16]).

More generally, if the measure μ is only quasi-invariant one can still define the Koopman type representation using Radon-Nikodim derivative:

$$(\kappa(g)f)(x) = \sqrt{\frac{\mathrm{d}\mu(g^{-1}(x))}{\mathrm{d}\mu(x)}}f(g^{-1}x).$$

If the measure μ is not invariant then constant functions do not form an invariant subspace and Koopman representation can be irreducible in the whole $L^2(X,\mu)$. There are several examples of group actions with quasi-invariant measures known for which the Koopman representation is irreducible:

1) actions of free non-commutative groups on their boundaries ([13], [14] and [26]);

2) actions of lattices of Lie-groups (or algebraic-groups) on their Poisson-Furstenberg boundaries ([9] and [6]);

3) action of the fundamental group of a compact negatively curved manifold on its boundary endowed with the Paterson-Sullivan measure class ([3]);

4) canonical actions of Higman-Thompson groups on segments endowed with Lebesgue measure ([15] and [10]).

However, the general case is not well understood and search for new examples is a challenging problem. In the present paper we construct a new class of examples of irreducible Koopman representations (see Theorem 3).

Observe that for any spherically homogeneous rooted tree and any $G < \mathrm{Aut}(T)$ the Koopman representation of G corresponding to the invariant probability measure μ on ∂T is a direct sum of countably many finite dimensional irreducible subrepresentations, since for each n the finite-dimensional subspace of functions constant on the subtrees emerging from vertices from V_n is invariant under G (see [4]). Recall that for two groups $G < H$ the centralizer of G in H is defined by:

$$C_H(G) = \{h \in H : hg = gh \text{ for all } g \in G\}.$$

Notice that for any group G acting on a rooted tree T we have the following group inclusions:

$$G < \mathrm{Aut}(T) < \mathrm{Aut}(\partial T, \mu) < \widetilde{\mathrm{Aut}}(\partial T, \mu) \text{ and} \tag{3}$$

$$\mathrm{Aut}(T) < \mathrm{Homeo}(\partial T) < \mathrm{Aut}(\partial T) < \mathrm{Bij}(T), \tag{4}$$

where $\mathrm{Homeo}(T)$ is the group of all homeomorphism of ∂T onto itself, $\widetilde{\mathrm{Aut}}(\partial T, \mu)$ is the group of all measure class preserving automorphisms of $(\partial T, \mu)$, $\mathrm{Aut}(\partial T)$ is the group of all Borel automorphisms of ∂T and $\mathrm{Bij}(T)$ is the group of all bijections of ∂T onto itself. One of the results of the present paper (Theorem 2) is related to the question formulated by Kosyak (see [24], Conjecture 0.0.1). For a measure space (X, ν) and a measure class preserving map $f : X \to X$ denote by $f_*(\nu)$ the push-forward measure on X. We will use the symbol Id for the identity operator.

Question 1 (Kosyak)**.** *In which cases for a measure preserving dynamical system (G, X, ν) irreducibility of the associated Koopman representation κ is equivalent to the following two conditions:*
1) *$f_*(\nu) \perp \nu$ for any $f \in C_{\widetilde{\mathrm{Aut}}(X,\nu)}(G), f \neq \mathrm{Id}$;*
2) *ν is G-ergodic.*

As a Corollary of Theorem 2 we obtain:

Corollary 2. *Let G be a weakly branch group acting on a spherically homogeneous rooted tree T. Then the centralizers of G in the groups* $\mathrm{Aut}(T), \mathrm{Homeo}(\partial T), \mathrm{Aut}(\partial T, \mu), \widetilde{\mathrm{Aut}}(\partial T, \mu)$ *and* $\mathrm{Aut}(\partial T)$ *are trivial.*

Here μ is the unique G-invariant probability measure on ∂T. Thus, for every weakly branch group the action of G on $(\partial T, \mu)$ satisfies conditions 1) and 2) of Question 1. Since the corresponding Koopman representation is reducible, this gives a class of dynamical systems for which conditions 1) and 2) are not sufficient to assure irreducibility of κ.

3 Stabilizers and centralizers of weakly branch group actions.

In this section we will prove Theorems 1 and 2. For simplicity, we denote ∂T by X and ∂T_v by X_v for a vertex v of T.

Lemma 1. *Let G be a weakly branch group acting on a spherically homogeneous rooted tree T. Then for every pair $x, y \in X$ of points from disjoint orbits the groups $\mathrm{St}_G(x)$ and $\mathrm{St}_G(y)$ are not quasi-conjugate in G.*

Proof. Assume that for some x, y from disjoint orbits the groups $\mathrm{St}_G(x)$ and $\mathrm{St}_G(y)$ are quasi-conjugate in G. By conjugating one of the groups by an appropriate element of g we can assume that $\mathrm{St}_G(x)$ and $\mathrm{St}_G(y)$ are commensurable (see Subsection 2.2). Equivalently, the orbits $\mathrm{St}_G(x)y$ and $\mathrm{St}_G(y)x$ are finite. Let us show that the latter is false.

Choose sufficiently large n such that for the vertices $v, w \in V_n$ with $x \in X_v$, $y \in X_w$ one has $v \neq w$. For any $k > n$ let $v_k \in V_k$ be the vertex such that $X_{v_k} \ni x$. Since G is weakly branch, there exists $g_k \in G, g_k \neq \mathrm{Id}$ (where Id is the trivial automorphism) such that $\mathrm{supp}(g_k) \subset X_{v_k}$. Then there exists $m > k$ and a vertex $u \in V_m$ such that $g_k u \neq u$. Since G is level transitive, there exists $h \in G$ such that $hu = v_m$. Set $h_k = hg_kh^{-1}$. Then $\mathrm{supp}(h_k) \subset X_v$, so in particular $h_k \in \mathrm{St}_G(y)$, and $h_k v_m \neq v_m$, and thus $h_k x \neq x$.

Finally, construct inductively an increasing sequence k_l such that $h_{k_l} x \notin X_{v_{k_{l+1}}}$ for every l. Then $h_{k_l} x$ are pairwise distinct, which shows that $\mathrm{St}_G(y)x$ is infinite. This contradiction finishes the proof of Lemma 1 ∎

As a corollary using Mackey Theorem 4 we obtain Theorem 1.

Proof of Theorem 2. Assume that the centralizer $\mathcal{C}$ of a weakly branch group G in $\mathrm{Bij}(X)$ is not trivial. Let $c \in \mathcal{C}$, $c \neq \mathrm{Id}$. Let $x \in X$ such that $cx \neq x$. Consider two cases:
a) $cx \in Gx$. Then c preserves the orbit Gx and the unitary operator $C : l^2(Gx) \to l^2(Gx)$ given by:

$$(Cf)(y) = f(c^{-1}y)$$

commutes with the representation ρ_x. Since ρ_x is irreducible, by Schur's Lemma (see e.g. [7], Proposition 2.2) C is a scalar operator. But this is impossible, since $C\delta_x = \delta_{cx}$ and δ_{cx} is orthogonal to δ_x.
b) $cx \notin Gx$. Then c maps the orbit Gx onto the orbit Gcx and the corresponding unitary operator $C : l^2(Gx) \to l^2(Gcx)$ intertwines representations ρ_x and ρ_{cx}, which is impossible since by Theorem 1 representations ρ_x and ρ_{cx} are disjoint. $\square$

4 Irreducibility of Koopman representations of weakly branch groups.

In this section we will prove Theorem 3. Let G be a weakly branch group acting on a rooted tree T, $X = \partial T$ and $p \in \mathcal{P}$ (see (1)). First we need to show that the Koopman representation corresponding to the action of G on (X, μ_p) is well-defined (i.e. that the measure μ_p is quasi-invariant).

Let T_n be the finite subtree of T composed from levels up to the nth. Observe that for each n the finite group of automorphism $\mathrm{Aut}(T_n)$ can be identified with a subgroup of $\mathrm{Aut}(T)$ consisting of elements g such that all the restrictions of g on subtrees of T emerging from vertices of n-th level are trivial. For $g \in \mathrm{Aut}(T)$ we denote by $g^{(n)} \in \mathrm{Aut}(T_n)$ the automorphism of T_n induced by g. We consider $g^{(n)}$ as an element of $\mathrm{Aut}(T)$.

Proposition 2. *For any subexponentially bounded automorphism g of a $d-$regular rooted tree T and any $p \in \mathcal{P}$ the measure μ_p on X is quasi-invariant with respect to g.*

Proof. Let $g \in \mathrm{Aut}(T)$ be subexponentially bounded and $p \in \mathcal{P}$. Denote by A_n the set of vertices $v \in V_n$ such that the restriction g_v is not equal to identity. Let $P = \max\{p_1, \ldots, p_d\}$. Set $M_n = \bigcup\limits_{v \in A_n} X_v$. Then by the definition

of subexponential boundedness one has:

$$\mu_p(M_n) \leqslant P^n |A_n| \to 0 \text{ when } n \to \infty.$$

Clearly, for every n the automorphism $g^{(n)}$ preserves the class of measure μ_p. Observe that on $X \setminus M_n$ the automorphism g coincides with $g^{(n)}$, and thus has well-defined Radon-Nikodym derivative on this set. It follows that g has well-defined Radon-Nikodym derivative for almost all $x \in X$ and hence is measure class preserving. $\square$

Proposition 3. *For any $p \in \mathcal{P}$ and any subexponentially bounded group G acting level-transitively on a d-regular rooted tree T the measure μ_p on X is ergodic with respect to the action of G.*

Proof. Assume that there exists a G-invariant subset $A \subset T$ such that $0 < \mu_p(A) < 1$. Let $v, w \in V_n$ for some $n \in \mathbb{N}$. Then there exists $g \in G$ such that $gv = w$. By subexponential boundedness of g for every $\epsilon > 0$ there exists $m > n$, $k \in \mathbb{N}$ and a collection $v_1, \ldots, v_k \in V_m$ such that the restriction of g on the subtree T_{v_i} is trivial for every $i = 1, \ldots, k$ and

$$\mu_p(B_m) > 1 - \epsilon, \quad \text{where } B_m = \bigcup_{i=1}^{k} X_{v_i}.$$

In particular, g coincides with $g^{(m)}$ on B_m and hence the Radon-Nikodym derivative of g is constant on X_{v_i} for every i. It follows that

$$\frac{\mu_p(g(A \cap X_{v_i}))}{\mu_p(A \cap X_{v_i})} = \frac{\mu_p(g(X_{v_i} \setminus A))}{\mu_p(X_{v_i} \setminus A)}.$$

Since $\epsilon > 0$ is arbitrary using G-invariance of A we obtain that

$$\frac{\mu_p(X_w \setminus A)}{\mu_p(A \cap X_w)} = \frac{\mu_p(g(X_v \setminus A))}{\mu_p(g(A \cap X_v))} = \frac{\mu_p(X_v \setminus A)}{\mu_p(A \cap X_v)}. \tag{5}$$

Since $v, w \in V_n$ are arbitrary, the latter is equal to $\frac{\mu_p(X \setminus A)}{\mu_p(A)} = \frac{1}{\mu_p(A)} - 1$. It follows that for every clopen set $B \subset X$ one has:

$$\mu_p(B \setminus A) = \left(\tfrac{1}{\mu_p(A)} - 1\right) \mu_p(A \cap B).$$

Recall that clopen subsets of X (which is homeomorphic to a Cantor set) approximate all measurable subsets by measure with arbitrary precision. Choosing a clopen subset B such that

$$\mu_p(A \cap B) > \tfrac{1}{2}\mu_p(A) \text{ and } \mu_p(B \setminus A) < \tfrac{1}{2}(1 - \mu_p(A))$$

we obtain a contradiction. This finishes the proof. $\square$

The proof of Theorem 3 is based on several technical statements. The following statement is a generalization of Proposition 23 from [11]. The proof is similar. For the reader's convenience we present it here.

Proposition 4. *Let G be a subexponentially bounded countable weakly branch group acting on a $d-$regular rooted tree T, $p \in \mathcal{P}^*$ (see (2)) and μ_p be the corresponding Bernoulli measure on X. For any clopen subset $A \subset X$ and any $\epsilon > 0$ there exists $g \in G$ such that*

$$\mathrm{supp}(g) \subset A \ \textit{and} \ \mu_p(A \setminus \mathrm{supp}(g)) < \epsilon.$$

Let us prove an auxiliary combinatorial lemma.

Lemma 2. *Let $n \in \mathbb{N}$ and let $H < \mathrm{Sym}(n)$ be a subgroup acting transitively on $\{1, 2, \ldots, n\}$. Let A be a subset of $\{1, \ldots, n\}$ such that for all $g \neq h \in H$ one has $|g(A)\Delta h(A)| \leqslant |A|$, where $|A|$ is the cardinality of A. Then $|A| > n/2$.*

Proof. Set $k = |A|$. Then for any two distinct elements $h, g \in H$ we have:

$$|h(A) \cap g(A)| = \frac{1}{2}(|h(A)| + |g(A)| - |h(A)\Delta g(A)|) \geqslant k/2.$$

For every $h \in H$ introduce a vector $\xi_h \in \mathbb{C}^n$ by:

$$\xi_h = (x_1, \ldots, x_n), \text{ where } x_i = \begin{cases} 0, & \text{if } i \notin h(A), \\ 1, & \text{if } i \in h(A). \end{cases}$$

Denote by $(\cdot,\cdot)$ the standard scalar product in $\mathbb{C}^n$ and by $\|\cdot\|$ the corresponding norm. Then for every $h \neq g \in H$ one has: $\|\xi_h\|^2 = k$ and $(\xi_h, \xi_g) = |h(A) \cap g(A)| \geqslant k/2$. We obtain:

$$\|\sum_{h \in H} \xi_h\|^2 \geqslant \tfrac{k}{2} m(m+1), \text{ where } m = |H|.$$

On the other hand, the group H acts on $\mathbb{C}^n$ by permuting coordinates such that $g(\xi_h) = \xi_{gh}$ for all $g, h \in H$. Using transitivity of H and the fact that the vector $\sum\limits_{h \in H} \xi_h$ is fixed by H we get:

$$\sum_{h \in H} \xi_h = (\tfrac{km}{n}, \tfrac{km}{n}, \dots, \tfrac{km}{n}), \quad \|\sum_{h \in H} \xi_h\|^2 = \tfrac{k^2m^2}{n}.$$

It follows that $km \geqslant n\frac{m+1}{2}$ and $k > \frac{n}{2}$. $\square$

Let $d \geqslant 2$ be the valency of the regular rooted tree T. The proof of Proposition 4 is based on the following:

Lemma 3. *If G is weakly branch then for every vertex v there exists $g \in G$ with* $\mathrm{supp}(g) \subset X_v$ *such that* $\mu_p(\mathrm{supp}(g)) \geqslant \frac{1}{d}\mu_p(X_v)$.

Proof. First, let us introduce some notations. For an element $h \in G$ set

$$l(h) = \max\{l : h \in \mathrm{St}_G(l)\}.$$

As before, for $h \in \mathrm{Aut}(T)$ and $n \in \mathbb{N}$ symbol $h^{(n)}$ denotes the element of $\mathrm{Aut}(T_n)$ induced by h. We will use the same symbol for the corresponding permutation of V_n. For a vertex $v \in V_n, n \in \mathbb{N}$ and $l \geqslant n$ set $V_l(v) = T_v \cap V_l$.

Fix a vertex v of the tree. Since G is weakly branch there exist $g \neq \mathrm{Id}$ such that $\mathrm{supp}(g) \subset X_v$. Set

$$L = \min\{l(g) : g \in G, g \neq \mathrm{Id}, \mathrm{supp}(g) \subset X_v\}.$$

For $g \in G$ denote by $W(g)$ the set of vertices w from $V_L(v)$ such that g induces a nontrivial permutation on $V_{L+1}(w)$. Set

$$k(g) = |W(g)|, \quad K = \max\{k(g) : g \in G, \mathrm{supp}(g) \subset X_v\}.$$

By the choice of L, $K > 0$. Fix an element $g \in G$ with $\mathrm{supp}(g) \subset X_v$ such that $k(g) = K$.

Further, since G acts transitively on V_L, we can find a number m and a collection of elements $H = \{h_1, h_2, \dots, h_m\} \subset G$ such that the family $S = \{h_1^{(L)}, \dots, h_m^{(L)}\}$ of transformations of V_L forms a group preserving $V_L(v)$ and transitive on $V_L(v)$. Denote $g_i = h_i g h_i^{-1}$. One has:

$$W(g_i) = h_i^{(L)}(W(g)), \quad W(g_i g_j) \supset W(g_i) \Delta W(g_j)$$

for all i, j. It follows that the set $W(g)$ together with the action of the group S restricted to $V_L(v)$ satisfy the conditions of Lemma 2. Therefore, $K = |W(g)| > \frac{1}{2}|V_L(v)|$.

For each $w \in W(g) \subset V_L(v)$ the element g induces a nontrivial permuation of $V_{L+1}(w)$, and thus there exists at least two vertexes $w_1, w_2 \in V_{L+1}(w)$ such that

$$\mathrm{supp}(g) \supset X_{w_1} \cup X_{w_2}.$$

Set $\mathcal{W} = \{w_1, w_2 : w \in W(g)\}$. We have:

$$|\mathcal{W}| > \tfrac{1}{d}|V_{L+1}(v)| \text{ and } \mathrm{supp}(g) \supset \bigcup_{u \in \mathcal{W}} X_u.$$

Now, since G is level transitive, we can find a number r and a collection of elements $\widetilde{H} = \{\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_r\} \subset G$ such that the family $\tilde{S} = \{\tilde{h}_1^{(L+1)}, \ldots, \tilde{h}_r^{(L+1)}\}$ of transformations of V_{L+1} forms a group preserving $V_{L+1}(v)$ and transitive on $V_{L+1}(v)$. Denote $\tilde{g}_i = \tilde{h}_i g \tilde{h}_i^{-1}$. One has:

$$\mathrm{supp}(\tilde{g}_i) \supset \bigcup_{u \in \mathcal{W}} X_{\tilde{h}_i^{(L+1)}(u)}.$$

Since $\widetilde{S}$ is a group acting transitively on $V_{L+1}(v)$, for every $u \in \mathcal{W}$ the multiset $\{\tilde{h}_i^{(L+1)}(u) : i = 1, \ldots, r\}$ contains every vertex $u_1 \in V_{L+1}(v)$ equal number $\frac{r}{|V_{L+1}(v)|}$ times. It follows that

$$\sum_{i=1}^{r} \mu_p(\mathrm{supp}(\tilde{g}_i)) \geqslant |\mathcal{W}| \frac{r}{|V_{L+1}(v)|} \mu_p(X_v) > \tfrac{r}{d}\mu_p(X_v).$$

Therefore, there exists i such that $\mu_p(\mathrm{supp}(\tilde{g}_i)) > \frac{1}{d}\mu_p(X_v)$. This finishes the proof. $\square$

Proof of Proposition 4. Let A be any clopen set and $g_0 = \mathrm{Id}$. Construct by induction elements $g_n \in G, n = 0, 1, 2 \ldots$ such that $\mathrm{supp}(g_n) \subset A$ and $\mu_p(A \setminus \mathrm{supp}(g_n)) \leqslant \left(\frac{d}{d+1}\right)^n$. If g_n is constructed choose vertices $v_1, \ldots v_k$ such that X_{v_j} are disjoint subsets of $A \setminus \mathrm{supp}(g_n)$ and $\sum \mu_p(X_{v_j}) \geqslant \frac{d}{d+1}\mu_p(A \setminus \mathrm{supp}(g_n))$. Using the lemma construct elements $h_1, \ldots h_k$ such that $\mathrm{supp}(h_j) \subset X_{v_j}$ and $\mu_p(\mathrm{supp}(h_j)) \geqslant \frac{1}{d}\mu_p(X_{v_j})$. Set $g_{n+1} = g_n h_1 h_2 \ldots h_k$. Then $\mu_p(A \setminus \mathrm{supp}(g_{n+1})) \leqslant \frac{d}{d+1}\mu_p(A \setminus \mathrm{supp}(g_n))$, which finishes the proof. $\square$

As before, let $\mathrm{St}_G(1) = \{g \in G : gv = v \quad \text{for each} \quad v \in V_1\}$ be the stabilizer of the first level of T. For each point $x \in X$ denote by $N_g(x) \in \mathbb{Z}_+ \cup \{\infty\}$ the number of the vertices v on the path defined by x such that $g_v \notin \mathrm{St}_G(1)$. Let T_n be the finite subtree of T composed from levels up to the nth. Observe that for each n the finite group of automorphism $\mathrm{Aut}(T_n)$ can be identified with a subgroup of $\mathrm{Aut}(T)$ consisting of elements g such that g_v is trivial for every $v \in V_n$. For $g \in \mathrm{Aut}(T)$ we denote by $g^{(n)} \in \mathrm{Aut}(T_n)$ the automorphism of T_n induced by g. We consider $g^{(n)}$ as an element of $\mathrm{Aut}(T)$.

Corollary 3. *For every clopen set A with $\mu_p(A) > 0$, any $\epsilon > 0$ and $k \in \mathbb{N}$ there exists $g \in G$ such that* $\mathrm{supp}(g) \subset A$ *and*

$$\mu_p(\{x : N_g(x) \geqslant k\}) > (1-\epsilon)\mu_p(A).$$

Proof. We prove the statement by induction on k. The base follows immediately from Proposition 4. Assume that for given clopen set A and $\epsilon > 0$ we have an element g such that

$$\mu_p(\{x : N_g(x) \geqslant k\}) > (1-\epsilon)\mu_p(A).$$

For n large enough one has $\mu_p(\{x : N_{g^{(n)}}(x) \geqslant k\}) > (1-\epsilon)\mu_p(A)$. Let

$$D = \{x : N_g(x) \geqslant k\},\ C = \{x : N_g(x) = k\},\ B = \{x \in C : N_{g^{(n)}}(x) = k\}.$$

Fix $\delta > 0$. Increasing n if necessary we can assume that $\mu_p(C \setminus B) < \delta$. Find a clopen set $\tilde{B} \subset A$ such that $\mu_p(B \Delta \tilde{B}) < \delta$. Using Proposition 4 find an element h with $\mathrm{supp}(h) \subset \tilde{B}$ such that $\mu_p(\tilde{B} \setminus \mathrm{supp}(h)) < \delta$ and $h \in \mathrm{St}_G(n+1)$. Then one has:

$$N_{gh}(x) \geqslant k+1 \ \text{ for all } \quad x \in S = (B \cap \mathrm{supp}(h)) \cup (D \setminus (C \cup \mathrm{supp}(h))).$$

By construction, $\mu_p(D \setminus S) < 4\delta$. It follows that for δ sufficiently small one has

$$\mu_p(\{x : N_{gh}(x) \geqslant k+1\}) \geqslant \mu_p(S) > (1-\epsilon)\mu_p(A).$$

$\square$

For $g \in \mathrm{Aut}(T)$ introduce an element $\alpha(g) \in \mathrm{Aut}(T)$ as follows. For arbitrary $x \in X$ let v be the vertex of a minimal level from the path defined by x such that $g(v) \neq v$. Write $x = vy$. Set

$$\alpha(g)x = g(v)y,$$

where $g(v)y$ should be understood as a concatenation of a finite word $g(v)$ and an infinite word y. Observe that for every $x \in X$ $\alpha(g)x$ differs with x at most by one letter. Define $\beta(g) = g\alpha(g)^{-1}$. Observe that for all x such that $gx \neq x$ one has

$$N_{\alpha(g)}(x) = 1, \quad N_{\beta(g)}(x) = N_g(x) - 1. \tag{6}$$

Recall that p_i are pairwise distinct. Set

$$a = a(p) = \min\{\tfrac{p_i}{p_j} : p_i > p_j\}, \quad \gamma = \gamma(p) = \frac{2\sqrt{a}}{a+1}.$$

Lemma 4. *Let A be a clopen set and ξ_A be the characteristic function of the set A. Let $g \in \mathrm{Aut}(T)$, $g(A) = A$ and $N_g(x) \geqslant k$ for almost all $x \in A$, where $k \in \mathbb{N}$. Then*

$$(\kappa_p(g)\xi_A, \xi_A) \leqslant \gamma^k \mu_p(A).$$

Proof. We will prove the lemma using induction by k. The base of induction $k = 0$ is trivial. Assume that the statement is true for given k. Let g be such that $N_g(x) \geqslant k+1$ for almost all $x \in A$. We will use the presentation $g = \alpha(g)\beta(g)$ (see (6)). Let v be any vertex such that $\alpha(g)v \neq v$ and $\alpha(g)w = w$ for the vertex w adjacent to v above v. Denote by $v_0 = v, v_1, \ldots, v_{l-1}$ the orbit of v under $\alpha(g)$, where $\alpha(g)v_j = v_{j+1}$ for $j < l-1$ and $\alpha(g)v_{l-1} = v_0$. Observe that $\beta(g)v_j = v_j$ for every j. From the induction assumption it follows that

$$(\kappa_p(\beta(g))\xi_{v_j}, \xi_{v_j}) \leqslant \gamma^k \mu_p(X_{v_j}).$$

For every j let q_j be the weight from $\{p_1, \ldots, p_d\}$ corresponding to the last letter of v_j. Then

$$\frac{\mathrm{d}\mu_p(\alpha(g)^{-1}(x))}{\mathrm{d}\mu_p(x)} = \frac{q_{j-1}}{q_j}$$

for all $x \in X_{v_j}$. We obtain:

$$\left(\kappa_p(g)\sum_{j=0}^{l-1}\xi_{v_j}, \sum_{j=0}^{l-1}\xi_{v_j}\right) = \sum_{j=0}^{l-1}\sqrt{\tfrac{q_{j-1}}{q_j}}(\kappa_p(\beta(g))\xi_{v_j}, \xi_{v_j}) \leqslant \gamma^k \sum_{j=0}^{l-1}\sqrt{\tfrac{q_{j-1}}{q_j}}\mu_p(X_{v_j}) =$$

$$\gamma^k \sum_{j=0}^{l-1}\sqrt{q_j q_{j-1}}\mu_p(X_w) \leqslant \gamma^{k+1}\sum_{j=0}^{l-1}\tfrac{q_j+q_{j-1}}{2}\mu_p(X_v) = \gamma^{k+1}\sum_{j=0}^{l-1}\mu_p(X_{v_j}).$$

Summing the last inequality over all the orbits $\{v_j\}$ as above (finite or countable number) we obtain the desired inequality. $\square$

For a unitary representation of a discrete group Γ in a Hilbert space H denote by $\mathcal{M}_\pi$ the von Neumann algebra generated by operators $\pi(g), g \in \Gamma$ (i.e. the closure of linear combinations of operators $\pi(g), g \in G$ in the weak operator topology). Let $B(H)$ be the algebra of all bounded operators on H and

$$\mathcal{M}'_\pi = \{R \in B(H) : QR = RQ \ \text{ for all } \ Q \in \mathcal{M}_\pi\}$$

be the commutant of $\mathcal{M}_\pi$. The following fact is folklore:

Lemma 5. *Let π be a unitary representation of a discrete group Γ in a Hilbert space H. Set $H_1 = \{\eta \in H : \pi(g)\eta = \eta \ \text{ for all } \ g \in \Gamma\}$. Then the orthogonal projection P onto H_1 belongs to $\mathcal{M}_\pi$.*

Proof. Let $B \in \mathcal{M}'_\pi$. Then

$$\pi(g)B\eta = B\pi(g)\eta = B\eta$$

for every $\eta \in H_1, g \in \Gamma$. This implies that $BH_1 \subset H_1$ and so $BP = PBP$. Same argument shows that $B^*P = PB^*P$, where * stands for the operation of conjugation in $B(H)$. Conjugating the latter identity we obtain that $PB = PBP = BP$. By von Neumann Bicommutant Theorem (see e.g. Theorem 2.4.11 in [8]) we get that $P \in (M'_\pi)' = \mathcal{M}_\pi$. $\square$

For an open set $A \subset X$ define

$$\begin{aligned} G_A &= \{g \in G : \mathrm{supp}(g) \subset A\}, \\ \mathcal{H}_A &= \{\eta \in \mathcal{H} : \pi(g)\eta = \eta \ \text{ for all } \ g \in G_A\}. \end{aligned} \tag{7}$$

Let P_A be the orthogonal projection onto $\mathcal{H}_A$. Applying Lemma 5 to the restriction of the representation κ_p onto the subgroup G_A we obtain

Corollary 4. *For any open subset $A \subset X$ one has $P_A \in \mathcal{M}_{\kappa_p}$.*

Proposition 5. *One has*

$$\mathcal{H}_A = \{\eta \in \mathcal{H} : \mathrm{supp}(\eta) \subset X \setminus A\}.$$

Proof. Clearly, every η with $\mathrm{supp}(\eta) \subset X \setminus A$ belongs to $\mathcal{H}_A$. Assume that $\mathcal{H}_A$ is strictly larger than the subspace of functions η with $\mathrm{supp}(\eta) \subset X \setminus A$. Then there exists a unit vector $\eta \in \mathcal{H}_A$ such that $\mathrm{supp}(\eta) \subset A$. Fix such a vector.

For $n \in \mathbb{N}$ denote by $V_n(A)$ the set of vertices v from V_n such that $X_v \subset A$. Let $\epsilon > 0$. Since locally constant functions are dense in $\mathcal{H}$, one can find a level n and constants $\alpha_v, v \in V_n(A)$ such that

$$\|\eta - \sum_{v \in V_n(A)} \alpha_v \xi_{X_v}\| \leqslant \epsilon.$$

By Corollary 3 and Lemma 4 for each $v \in V_n(A)$ there exists a sequence of elements $g_{v,k}$ such that

$$\mathrm{supp}(g_{v,k}) \subset X_v \text{ and } \lim_{k\to\infty} (\pi(g_{v,k})\xi_{X_v}, \xi_{X_v}) = 0.$$

Set $h_k = \prod_{v \in V_n(A)} g_{v,k}$. Then $\mathrm{supp}(h_k) \subset A$ and

$$\lim_{k\to\infty} (\pi(h_k) \sum_{v \in V_n(A)} \alpha_v \xi_{X_v}, \sum_{v \in V_n(A)} \alpha_v \xi_{X_v}) = 0.$$

It follows that

$$\limsup_{k\to\infty} |(\pi(h_k)\eta, \eta)| \leqslant 2\epsilon + \epsilon^2.$$

Taking ϵ, for instance, to be $\frac{1}{3}$ we obtain a contradiction to the fact that $\eta \in \mathcal{H}_A$ is a unit vector. This finishes the proof. $\square$

Proof of Theorem 3. 1) Proposition 5 means that for each open set $A \subset X$ the orthogonal projection P_A onto $\mathcal{H}_A$ is the operator of multiplication by the characteristic function of $X \setminus A$. Observe that every function from $L^\infty(X, \mu_p)$ can be approximated arbitrarily well in L^2-norm by finite linear combinations of characteristic functions of open sets. This implies that for every $m \in L^\infty(X, \mu_p)$ the operator of multiplication by m

$$\mathcal{H} \to \mathcal{H}, \;\; f \to mf$$

can be approximated arbitrary well in the strong operator topology by finite linear combinations of projections $P_A \in \mathcal{M}_{\kappa_p}$ (see Corollary 4), and thus belongs to the von Neumann algebra $\mathcal{M}_{\kappa_p}$ generated by operators $\kappa_p(g), g \in G$. This implies that $\mathcal{M}_{\kappa_p}$ contains operators of multiplication by all functions from $L^\infty(X, \mu_p)$. Since for an ergodic measure class preserving action of a group G on a measure space (X, μ_p) the algebra generated by group shifts and multiplication by functions coincide with the algebra $B(\mathcal{H})$ of all bounded

operators on $\mathcal{H}$ (see e.g. [31], Corollary 1.6) we obtain that $\mathcal{M}_{\kappa_p}$ coincides with $B(\mathcal{H})$. By Schur's Lemma (see e.g. [7], Theorem A.2.2) this implies irreducibility of κ_p.

2) Let $x \in X$. For an open subset A of X denote by $\mathcal{H}_A^x$ the subspace of $l^2(Gx)$ analogous to (7), but corresponding to the representation ρ_x:

$$\mathcal{H}_A^x = \{\eta \in l^2(Gx) : \rho_x(g)\eta = \eta \ \text{ for all } \ g \in G_A\}.$$

Let P_A^x be the orthogonal projection onto $\mathcal{H}_A^x$. Assume that κ_p and ρ_x are unitary equivalent via intertwining isometry

$$U : L^2(X, \mu_p) \to l^2(Gx),$$

that is $U\kappa_\pi(g) = \rho_x(g)U$ for every $g \in G$. Choose a sequence of open covers A_n of the orbit Gx such that $\mu_p(A_n) \to 0$ when $n \to \infty$. From the definition of orthogonal projections P_A and subspaces $\mathcal{H}_A$ we obtain:

$$UP_{A_n}U^* = P_{A_n}^x$$

for every n. Since G is weakly branch for every n and every $y \in Gx$ the set $\{gy : g \in G_{A_n}\}$ is infinite. This implies that $P_{A_n}^x = 0$. However, Proposition 5 implies that $P_{A_n} \to \mathrm{Id}$ weakly when $n \to \infty$. This contradiction shows that κ_p and ρ_x are disjoint (not unitary equivalent).

3) Let $\tilde{p} \in \mathcal{P}^*, \tilde{p} \neq p$. For an open subset A of X denote by $\tilde{\mathcal{H}}_A$ the subspace of $L^2(X, \mu_{\tilde{p}})$ analogous to (7), but corresponding to the representation $\kappa_{\tilde{p}}$. Let $\tilde{P}_A$ be the orthogonal projection onto $\tilde{\mathcal{H}}_A$. Since $\mu_{\tilde{p}}$ is singular to μ_p there exists $A \subset X$ such that $\mu_p(A) = 0$ and $\mu_{\tilde{p}}(A) = 1$. Assume that κ_p and $\kappa_{\tilde{p}}$ are unitary equivalent via intertwining isometry

$$U : L^2(X, \mu_p) \to L^2(X, \mu_{\tilde{p}}).$$

Let A_n be a sequence of open covers of A such that $\mu_p(A_n) \to 0$ when $n \to \infty$. Since $A \subset A_n$ we have that $\mu_{\tilde{p}}(A_n) = 1$ for every n. From the definition of orthogonal projections P_A and subspaces $\mathcal{H}_A$ we obtain:

$$UP_{A_n}U^* = \tilde{P}_{A_n} = \mathrm{Id}$$

for every n. But from Proposition 5 we obtain that $P_{A_n} \to 0$ weakly when $n \to \infty$. This contradiction finishes the proof. □

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